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Analysis of a Monod–Haldene type food chain chemostat with seasonally variably pulsed input and washout

Fengyan Wang*

College of Science, Jimei University, Xiamen, Fujian 361021, P.R. China
E-mail: wangfy68@163.com

Guoping Pang

Department of Mathematics and Computer Science, Yulin Normal University, Yulin, Guangxi 537000, P.R. China

Jin Hui

Department of Information and Computation Sciences, Guangxi University of Technology, Liuzhou, Guangxi 545006, P.R. China

In this paper, we introduce and study a model of a Monod–Haldene type food chain chemostat with seasonally variably pulsed input and washout. We investigate the subsystem with substrate and prey and study the stability of the periodic solutions, which are the boundary periodic solutions of the system. The stability analysis of the boundary periodic solution yields an invasion threshold. By use of standard techniques of bifurcation theory, we prove that above this threshold there are periodic oscillations in substrate, prey and predator. Simple cycles may give way to chaos in a cascade of period-doubling bifurcations. Furthermore, bifurcation diagrams have shown that there exists complexity for the pulsed system including periodic doubling cascade, periodic halving cascade and Pitchfork bifurcations and tangent bifurcations.

KEY WORDS: Monod–Haldene growth rate, chemostat, seasonally variably pulsed input and washout, chaos.

AMS subject classification: 34C35, 92D25

1. Introduction and the model

A chemostat is a common laboratory apparatus used to culture microorganisms. Sterile growth medium enters the chemostat at a constant rate; the volume within the chemostat is held constant. In its simplest form, the system approximates conditions for plankton growth in lakes, where the limiting nutrients such as silica and phosphate are supplied from streams draining the water-

*Corresponding author.

shed. As seasons change, stream drainage patterns change causing variations in the supply of nutrients and the washout of lakes. Recently many papers studied chemostat model with variations in the supply of nutrients or the washout. Chemostat with periodic inputs are studied in [1–5], those with periodic washout rate in [6, 7], and those with periodic input and washout in [8]. We all know that nutrients are inputted into lakes and lakes are washed out when rain is falling. In fact, raining is not continuous. It occurs seasonally or in regular pulses. Thus, it is natural to describe this case in impulsive differential equations. In the present paper, we consider the dynamics of a bi-trophic food chain model in a chemostat seasonally variably pulsed input and washout, which incorporate the Monod–Haldene type growth rate. Without loss of generality, we assume that the rains occur seasonally at k-times ($k \in N$) in every year T. The model takes the form:

$$\begin{cases} \left.\begin{aligned} \frac{\mathrm{d}S}{\mathrm{d}t} &= -\frac{\mu_1}{\delta_1}\frac{SH}{(A_1+S+B_1S^2)}, \\ \frac{\mathrm{d}H}{\mathrm{d}t} &= \frac{\mu_1 SH}{A_1+S+B_1S^2} - \frac{\mu_2}{\delta_2}\frac{HP}{(A_2+H+B_2H^2)}, \\ \frac{\mathrm{d}P}{\mathrm{d}t} &= \frac{\mu_2 HP}{A_2+H+B_2H^2}, \end{aligned}\right\} & t \neq nT+T_i, \; (i=1,2,\ldots,k; n \in N) \\ \left.\begin{aligned} \triangle S &= D_i(S_0 - S), \\ \triangle H &= -D_i H, \\ \triangle P &= -D_i P, \end{aligned}\right\} & t = nT+T_i, \; (i=1,2,\ldots,k; n \in N), \end{cases} \tag{1.1}$$

where T is the period of the impulsive effect and $0 < T_1 < T_2 < \cdots < T_k = T$ are the times of the impulsive effects in per period T. where $S(t)$ denotes the concentration of nutrient at time t; H(t) denotes the concentration of prey at time t; P(t) denotes the concentration of predator at time t; S_0 denotes the input concentration of the nutrient each time; $0 < D_i < 1$ $(i = 1, 2, \ldots, k)$ are the washout proportion of the chemostat each time $nT+T_i$, respectively; δ_1 and δ_2 denote the yield constants of unit mass of prey and predator; b_1, b_2 are half capturing saturation constants of prey and predator; μ_1 and m_2 denote the predation constants of prey and predator, respectively. $n \in N$, N is the set of all non-negative integers.

The theory of impulsive differential equation appears as a natural description of several real processes subject to certain perturbations whose duration is negligible in comparison with the duration of the process. Recently, equations of this kind are found in a almost every domain of applied sciences. Numerous examples are given in Bainov's and his collaborator's books [9, 10]. Some impulsive differential equations have been recently introduced in population dynamics in relation to: impulsive birth [11], impulsive vaccination [12, 13], chemotherapeutic treatment of disease [14] and population ecology [15–17].

Recently the models of predator-prey system in periodic forcing and impulsive effect environments have attracted new attention because of their propensity for chaos ([15–17]). Liu and Chen [15], Zhang et al. [16] have studied predator-prey system with Holling type-II [15] and type-IV [16] with impulsive perturbations on the predator, and the impulsive perturbations bring to the system complexity. Wang et al. [17] studied the three food chain with impulsive effects on top predator, and the impulsive perturbations also bring to these simple system chaotic solutions. In this paper, we want to investigate the complexity of system (1.1).

There are advantages in analyzing dimensionless equations. We take variable changes as following

$$x \equiv \frac{S}{S_0}, \quad y \equiv \frac{H}{\delta_1 S_0}, \quad z \equiv \frac{P}{\delta_1 \delta_2 S_0}.$$

After some algebra, this yields

$$\begin{cases} \left.\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= -\frac{m_1 x y}{1 + a_1 x + b_1 x^2}, \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= \frac{m_1 x y}{1 + a_1 x + b_1 x^2} - \frac{m_2 y z}{1 + a_2 y + b_2 y^2}, \\ \frac{\mathrm{d}z}{\mathrm{d}t} &= \frac{m_2 y z}{1 + a_2 y + b_2 y^2}, \end{aligned}\right\} & t \neq n\tau + \tau_i, (i = 1, 2, \ldots, k), \\ \left.\begin{aligned} \triangle x &= d_i(1 - x), \\ \triangle y &= -d_i y, \\ \triangle z &= -d_i z. \end{aligned}\right\} & t = n\tau + \tau_i, (i = 1, 2, \ldots, k). \end{cases} \tag{1.2}$$

with

$$m_1 = \frac{\mu_1 S^0}{A_1}, \quad m_2 = \frac{\mu_2 S^0}{A_2}, \quad a_1 = \frac{S^0}{A_1}, \quad a_2 = \frac{S^0}{A_2}, \quad b_1 = \frac{B_1 S_0^2)}{A_1},$$

$$b_2 = \frac{B_2 \delta_1^2 S_0^2}{A_2}, \quad \tau = T, \qquad d_i = D_i, \quad \tau_i = T_i, \quad (i = 1, 2, \ldots, k).$$

For convenance, through this paper, we denote $d_0 = 0$ and $\tau_0 = 0$.

The organizations of the paper are as following. In next section, we investigate the existence and stability of the periodic solutions of the impulsive subsystem with substrate and prey. In Section 3, we study the locally stability of the boundary periodic solution of the system and obtain the threshold of the invasion of the predator. By use of standard techniques of bifurcation theory, we prove that above this threshold there are periodic oscillations in substrate, prey and predator. In Section 4, the bifurcation diagrams of different coefficients show that with increasing the bifurcation parameters, there exists complexity for the pulsed system including periodic doubling cascade, periodic halving cascade and Pitchfork bifurcations and tangent bifurcations.

2. Behavior of the substrate bacterium subsystem

In the absence of the protozan predator, system (1.2) reduces to

$$\begin{cases} \left.\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= -\frac{m_1xy}{1+a_1x+b_1x^2}, \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= \frac{m_1xy}{1+a_1x+b_1x^2} \end{aligned}\right\} & t \neq n\tau+\tau_i, (i=1,2,\dots,k), \\ \left.\begin{aligned} \triangle x &= d_i(1-x), \\ \triangle y &= -d_i y. \end{aligned}\right\} & t = n\tau+\tau_i, (i=1,2,\dots,k). \end{cases} \tag{2.1}$$

This non-linear system has simple periodic solutions. For our purpose, we present these solutions in this sections.

If we add the first and second equations of the system (2.1), we have $\frac{\mathrm{d}(x+y)}{\mathrm{d}t} = 0$. If we take variable changes $s = x + y$ then the system (2.1) can be rewritten

$$\begin{cases} \frac{\mathrm{d}s}{\mathrm{d}t} = 0, & t \neq n\tau+\tau_i, (i=1,2,\dots,k), \\ s(t^+) = d_i + (1-d_i)s(t), s(0) > 0, & t = n\tau+\tau_i, (i=1,2,\dots,k). \end{cases} \tag{2.2}$$

For the system (2.2), we have the following lemma 2.1.

Lemma 2.1. The subsystem (2.2) has a positive periodic solution $\tilde{s}(t) = 1$ and for every solution $s(t)$ of (2.2) we have $|s(t)-1| \to 0$ as $t \to \infty$, where $\tilde{s}(t) = 1, t \in (n\tau, (n+1)\tau]$, $n \in N$.

By the lemma 2.1, the following lemma is obvious.

Lemma 2.2. Let $(x(t), y(t))$ be any solution of system (2.1) with initial condition $x(0) \geqslant 0$, $y(0) > 0$, then $\lim_{t\to\infty} |x(t)+y(t)-1| = 0$.

The lemma 2.2 says that the periodic solution $\tilde{s}(t) = 1$ is uniquely invariant manifold of the system (2.1).

Theorem 2.1. For the system (2.1), we denote

$$m_1^* := \frac{-(1+a_1+b_1)\sum_{i=1}^{k}\ln(1-d_i)}{\tau}.$$

(1) If $m_1 < m_1^*$, then the system (2.1) has a unique globally asymptotically stable positive τ-periodic solution $(x_e(t), y_e(t))$, where

$$x_e(t) = 1,\, y_e(t) = 0; \tag{2.3}$$

(2) If $m_1 > m_1^*$, then the system (2.1) has a unique globally asymptotically stable positive $\tau-$periodic solution $(x_s(t), y_s(t))$ and the τ-periodic solution $(x_e(t), y_e(t))$ is unstable. The τ-period positive solution $y_s(t)$ satisfies

$$\exp(\int_0^\tau \frac{m_1(1-y_s(l))}{1+a_1(1-y_s(l))+b_1(1-y_s(l))^2}\mathrm{d}l) = \prod_{i=1}^k \frac{1}{1-d_i}. \tag{2.4}$$

Proof. By lemma 2.1, we can consider the system (2.1) in its stable invariant manifold $\tilde{s}(t) = 1$, that is

$$\begin{cases} \dfrac{\mathrm{d}y}{\mathrm{d}t} = \dfrac{m_1(1-y)y}{1+a_1(1-y)+b_1(1-y)^2}, & t \neq n\tau + \tau_i,\, (i = 1, 2, \ldots, k), \\ \triangle y = -d_i y,\, 0 < y_0 \leqslant 1, & t = n\tau + \tau_i,\, (i = 1, 2, \ldots, k). \end{cases} \tag{2.5}$$

Suppose $y(t, y_0)$ is a solution of equation (2.5), with initial condition $y_0 \in [0, 1]$. We have

$$\begin{array}{ll} y(t, y_0) = y((n\tau+\tau_i)^+) & \\ \exp\left(\displaystyle\int_{n\tau+\tau_i}^t \frac{m_1(1-y(l, y_0))}{1+a_1(1-y(l, y_0))+b_1(1-y(l, y_0))^2}\mathrm{d}l\right), & t \in (n\tau+\tau_i, n\tau+\tau_{i+1}], \\ y((n\tau+\tau_i)^+) = (1-d_i)y(n\tau+\tau_i), \quad y(0^+) = y_0, & t = n\tau+\tau_i, \end{array} \tag{2.6}$$

For (2.5), we have the following properties:

(1) $0 < y(t, y_0) < 1, t \in (0, \infty)$ is piecewise continuous function;

(2) The function $F(y_0) = y(t, y_0), y_0 \in (0, 1]$ is a increasing function;

(3) $y(t, 0) = 0, t \in (0, \infty)$ is a solution .

The periodic solutions of (2.5) satisfy the following equation

$$y_0 = y_0 \prod_{i=1}^k (1-d_i) \exp\left(\int_0^\tau \frac{m_1(1-y(l, y_0))}{1+a_1(1-y(l, y_0))+b_1(1-y(l, y_0))^2}\mathrm{d}l\right). \tag{2.7}$$

By (i) and (ii), we know that if $1 < \prod_{i=1}^k (1-d_i) < \exp(\frac{m_1\tau}{1+a_1+b_1})$, that is $m_1 < m_1^*$, the equation (2.5) has a unique solution in (0, 1]; otherwise, it has no solution in (0, 1].

If $m_1 < m_1^*$, then the equation (2.5) has stable periodic solution $y_e(t) = 0$. By lemma 2.2, we have $\lim_{t\to\infty} |x(t) - \tilde{s}(t)| = 0$. We have proved in (1).

If $m_1 > m_1^*$, then the equation (2.5) has uniquely positive periodic solution. We denote this positive periodic solution

$$y_s(t) = y(t, y_0^*), \quad x_s(t) = \tilde{s}(t) - y(t, y_0^*).$$

From (2.6), we obtain the formate (2.4) hold.

For proving the stability of the period solution $y_s(t)$, we define a function $G(y_0)\colon y_0 \in (0, 1)$ as following:

$$G(y_0) = \prod_{i=1}^{k}(1-d_i)\exp\left(\int_0^{\tau} \frac{m_1(1-y(l, y_0))}{a_1 + (1-y(l, y_0)) + b_1(1-y(l, y_0))^2}\,\mathrm{d}l\right).$$

Noticing equation (2.5), we have

$$G(y_0) = \frac{y(\tau, y_0)}{y_0}, \quad y_0 \in (0, 1). \tag{2.8}$$

It is obvious that $G(y_0^*)) = 1$.

Furthermore, $\frac{\partial y(t, y_0)}{\partial y_0} \geqslant 0, t \in (0, \tau)$ is hold (otherwise, there exist $t_0 > 0, 0 < y_1 < y_2 < 1$ such that $y(t_0, y_1) = y(t_0, y_2)$, that is a contradiction with the different flows of system (2.5) not to intersect). So we obtain that the function $G(y_0)$ have the following properties:

$$\begin{aligned} &G(y_0) < 1, \quad if \quad y_0^* < y_0 < 1,\\ &G(y_0) = 1, \quad if \quad y_0 = y_0^*,\\ &G(y_0) > 1, \quad if \quad 0 < y_0 < y_0^*. \end{aligned} \tag{2.9}$$

Furthermore, we obtain the following equations

$$\begin{array}{ll} y_0 > y(\tau_1, y_0) > \cdots > y(n\tau + \tau_i, y_0) > y_0^*, & \text{if} \quad y_0^* < y_0 \leqslant 1,\\ y_0 < y(\tau_1, y_0) < \cdots < y(n\tau + \tau_i, y_0) < y_0^*, & \text{if} \quad \varepsilon \leqslant y_0 < y_0^*. \end{array} \tag{2.10}$$

Set $y_0 \in (0, 1)$. According to (2.10), we suppose that

$$\lim_{n\to\infty} y(n\tau, y_0) = a.$$

We shall prove that the solution $y(t, a)$ is τ-periodic. We note that the functions $y_n(t) = y(t + n\tau, y_0)$, due to the τ-periodicity of equation (2.5), are also its solutions and $y_n(0) \to a$ as $n \to \infty$. By the continuous dependence of the solutions on the initial values we have that $y(\tau, a) = \lim_{n\to\infty} y_n(\tau) = a$. Hence the solution $y(t, a)$ is τ-periodic. The periodic solution $y(t, y_0^*)$ is unique, so $a = y_0^*$.

Let $\varepsilon > 0$ be given. By the theorem 2.9 [9] on the continuous dependence of the solutions on the initial values, there exists a $\delta > 0$ such that

$$|y(t, y_0) - y(t, y_0^*)| < \varepsilon,$$

if $|y_0 - y_0^*| < \delta$ and $0 \leqslant t \leqslant \tau$. Choose $n_1 > 0$ so that $|y(n\tau, y_0) - y_0^*| < \delta$ for $n > n_1$. Then $|y(t, y_0) - y(t, y_0^*)| < \varepsilon$ for $t > n\tau$ which proves that

$$\lim_{n\to\infty} |y(t, y_0) - y(t, y_0^*)| = 0, \quad y_0 \in (0, \tilde{s}(0)].$$

For the system (2.1), by lemma 2.2 we obtain that for any solution $(x(t), y(t))$ with initial condition $x(0) \geqslant 0, y(0) > 0$, $|x - x_s| \to 0, |y - y_s| \to 0$ as $t \to \infty$.

From the τ-period solution y_s being globally asymptotically stable, we can obtain that the multiplier μ of y_s, which satisfies

$$\mu = \exp\left(\int_0^\tau \frac{m_1 x_s(l)(1 - b_1 x_s^2(l))}{1 + a_1 x_s(l) + b_1 x_s^2(l)} \mathrm{d}l\right) < 1, \tag{2.11}$$

where we have used (2.7). This conclusion will be used in the Section 3. We have proved (2). □

3. Stability of the boundary periodic solution

In order to investigate the invasion of the predator of system (1.2), we add the first, second and third equations of it and take variable changes $s = x+y+z$, then we obtain the following system

$$\begin{cases} \dfrac{\mathrm{d}s}{\mathrm{d}t} = 0, & t \neq n\tau + \tau_i, (i = 1, 2, \ldots, k), \\ s(t^+) = d_i + (1 - d_i)s(t), s(0) > 0, & t = n\tau + \tau_i, (i = 1, 2, \ldots, k). \end{cases}$$

By the lemma 2.1, the following lemma is obvious.

Lemma 3.1. Let $(x(t), y(t), z(t))$ be any solution of system (1.2) with $X(0) > 0$, then

$$\lim_{t\to\infty} |x(t) + y(t) + z(t) - 1| = 0. \tag{3.1}$$

The lemma 3.1 says that the periodic solution $\tilde{s}(t) = 1$ is an invariant manifold of the system (1.2).

By theorem 2.1, we know that the system system (1.2) has the nonnegative boundary τ-period solutions

$$(x_e(t), y_e(t), 0) = (1, 0, 0), \quad (x_s(t), y_s(t), 0) \ (if \ m_1 > m_1^*).$$

For convenance, in the following discussing if $m_1 > m_1^*$, we denote that

$$m_2^* := \frac{-\sum_{i=1}^k (1 - d_i)}{\int_0^\tau \frac{y_s(l)}{1+a_2 y_s(l)+b_2 y_s^2(l)} \mathrm{d}l}.$$

Theorem 3.1. Let $(x(t), y(t), z(t))$ be any solution of system (1.2) with $X(0) > 0$.

(1) If $m_1 < m_1^*$, then the system (3.1) has a unique globally asymptotically stable positive τ-periodic solution $(1, 0, 0)$.

(2) If $m_1 > m_1^*$ and $m_2 < m_2^*$, then the system (1.2) has a unique globally asymptotically stable boundary τ-periodic solution $(x_s(t), y_s(t), 0)$ is globally asymptotical stable.

(3) If $m_1 > m_1^*$ and $m_2 > m_2^*$, then the periodic boundary solution $(1 - y_s(t), y_s(t), 0)$ of the system (1.2) is unstable.

Proof. The proof of (1) is easy, we want to prove (2) and (3). The local stability of periodic solution $(x_s(t), y_s(t), 0)$ may be determined by considering the behavior of small amplitude perturbations of the solution. Define

$$x(t) = u(t) + x_s(t), y(t) = v(t) + y_s(t), z(t) = w(t),$$

there may be written

$$\begin{pmatrix} u(t) \\ v(t) \\ w(t) \end{pmatrix} = \Phi_i(t) \begin{pmatrix} u(\tau_{i-1}^+) \\ v(\tau_{i-1}^+) \\ w(\tau_{i-1}^+) \end{pmatrix} \qquad \tau_{i-1} < t \leqslant \tau_i, (i = 1, 2, \ldots, k),$$

where $\Phi_i(t)$ satisfies

$$\frac{\mathrm{d}\Phi_i}{\mathrm{d}t} = \begin{pmatrix} -\dfrac{m_1 y_s(1 - b_1 x_s^2)}{(1 + a_1 x_s + b_1 x_s^2)^2} & -\frac{m_1 x_s}{1+a_1 x_s+b_1 x_s^2} & 0 \\ \dfrac{m_1 y_s(1 - b_1 x_s^2)}{(1 + a_1 x_s + b_1 x_s^2)^2} & \frac{m_1 x_s}{1+a_1 x_s+b_1 x_s^2} & -\frac{m_2 y_s}{1+a_2 y_s+b_2 y_s^2} \\ 0 & 0 & \frac{m_2 y_s}{1+a_2 y_s+b_2 y_s^2} \end{pmatrix} \Phi_i(t),$$

and $\Phi_i(\tau_{i-1}^+) = I$, the identity matrix. Hence the fundamental solution matrix is

$$\Phi_i(\tau_i) = \begin{pmatrix} \phi_{1i}(\tau_i) & \phi_{2i}(\tau_i) & * \\ \phi_{3i}(\tau_i) & \phi_{4i}(\tau_i) & ** \\ 0 & 0 & \exp\left(\int_{\tau_{i-1}}^{\tau_i} \frac{m_2 y_s(l)}{1+a_2 y_s(l)+b_2 y_s^2(l)} \mathrm{d}l\right) \end{pmatrix}. \tag{3.2}$$

It is no need to give the exact form of $(*)$ and $(**)$ as it is not required in the analysis that follows. The linearization of impulsive subsystem (1.2) become

$$\begin{pmatrix} u(n\tau_i^+) \\ v(n\tau_i^+) \\ w(n\tau_i^+) \end{pmatrix} = \begin{pmatrix} 1 - d_i & 0 & 0 \\ 0 & 1 - d_i & 0 \\ 0 & 0 & 1 - d_i \end{pmatrix} \begin{pmatrix} u(n\tau_i) \\ v(n\tau_i) \\ w(n\tau_i) \end{pmatrix}.$$

We denote that

$$M_i = \begin{pmatrix} (1-d_i) & 0 & 0 \\ 0 & (1-d_i) & 0 \\ 0 & 0 & (1-d_i) \end{pmatrix} \Phi_i(\tau_i), (i = 1, 2, \dots, k).$$

Hence, we obtain the fundamental solution matrix M is

$$\begin{aligned} M &= M_k \cdots M_2 M_1 \\ &= \begin{pmatrix} \phi_{11}(\tau) & \phi_{12}(\tau) & * \\ \phi_{21}(\tau) & \phi_{22}(\tau) & ** \\ 0 & 0 & \prod_{i=1}^{k}(1-d_i)\exp\left(\int_0^\tau \dfrac{m_2 y_s(l)}{1+a_2 y_s(l)+b_2 y_s^2(l)} dl\right) \end{pmatrix}. \end{aligned}$$

The eigenvalues of the matrix M are $\mu_3 = \prod_{i=1}^{k}(1-d_i)\exp\left(\int_0^\tau (\frac{m_2 y_s(l)}{1+a_2 y_s(l)+b_2 y_s^2(l)} \mathrm{d}l\right)$ and the eigenvalues μ_1, μ_2 of the following matrix

$$\begin{pmatrix} \phi_{11}(\tau) & \phi_{12}(\tau) \\ \phi_{21}(\tau) & \phi_{22}(\tau) \end{pmatrix}. \tag{3.3}$$

The μ_1, μ_2 are also the multipliers the locally linearizing system of the system (2.1) provided with $m > m_1^*$ at the asymptotically stable periodic solution $(x_s(t), y_s(t))$, according to Theorem 2.1, we have that $\mu_1 = \mu_2 = \mu < 1$.

If $m_2 < m_2^*$, the $\prod_{i=1}^{k}(1-d_i)\exp(\int_0^\tau \frac{m_2 y_s(l)}{1+a_2 y_s(l)} \mathrm{d}l) < 1$, the boundary periodic solution $(x_s(t), y_s(t), 0)$ of the system (1.2) is locally asymptotically stable. We have that

$$\begin{aligned} z(t) &= z_0\Big(\prod_{j=1}^{k}(1-d_j)\Big)^n \prod_{j=1}^{i}(1-d_j)\Big)\exp\left(\int_0^t \frac{m_2 y_s(l)}{1+a_2 y_s(l)+b_2 y_s^2(l)} \mathrm{d}l\right), \\ & t \in (n\tau + \tau_i, n\tau + \tau_{i+1}], \quad i = 1, 2, \dots, k. \end{aligned}$$

Hence we obtain that for any solution $(x(t), y(t), z(t))$ with $X(0) > 0$, $z(t) \to 0$ as $t \to \infty$. By $\lim_{t\to\infty} |x(t) + y(t) + z(t) - \tilde{s}(t)| = 0$, we have $\lim_{t\to\infty} |x(t) + y(t) - \tilde{s}(t)| = 0$. Now using theorem 2.1, we have $\lim_{t\to\infty} |y(t) - y_s(t)| = 0$ and $\lim_{t\to\infty} |x(t) - x_s(t)| = 0$.

If $m_2 > m_2^*$, the $(\prod_{i=1}^{k}(1-d_i))\exp(\int_0^\tau \frac{m_2 y_s(l)}{1+a_2 y_s(l)+b_2 y_s^2(l)} \mathrm{d}l) > 1$, the boundary periodic solution $(x_s(t), y_s(t), 0)$ of the system (1.2) is unstable. We complete the proof. □

Let B denote the Banach space of piecewise continuous, τ-periodic functions $N: [0, \tau] \to R^2$ and have points of discontinuity τ_i, $(i = 1, 2, \dots, k)$, where they are continuous from the left. In the set B introduce the norm $|N|_0 = \sup_{0\leqslant t\leqslant\tau} |N(t)|$ with which B becomes a Banach space with the uniform convergence topology.

For convenience, just like [18] we introduce the following lemma 3.2 and 3.3.

Lemma 3.2. Suppose $a_{ij} \in B$ and $0 \leqslant d_{1i} < 1, 0 \leqslant d_{2i} < 1, (i = 1, 2, \ldots, k)$.

(a) If $\prod_{i=1}^{k}(1-d_{2i})\exp(\int_0^\tau a_{22}(s)\mathrm{d}s) \neq 1$, $\prod_{i=1}^{k}(1-d_{1i})\exp(\int_0^\tau a_{11}(s)\mathrm{d}s) \neq 1$, then the linear impulsive homogenous system

$$\begin{cases} \left.\begin{array}{l} \frac{\mathrm{d}y_1}{\mathrm{d}t} = a_{11}y_1 + a_{12}y_2, \\ \frac{\mathrm{d}y_2}{\mathrm{d}t} = a_{22}y_2, \end{array}\right\} & t \neq n\tau + \tau_i, (i = 1, 2, \ldots, k), \\ \left.\begin{array}{l} \triangle y_1 = -d_{1i}y_1, \\ \triangle y_2 = -d_{2i}y_2, \end{array}\right\} & t = n\tau + \tau_i, (i = 1, 2, \ldots, k). \end{cases} \tag{3.4}$$

has no nontrivial solution in $B \times B$. In this case the nonhomogeneous system

$$\begin{cases} \left.\begin{array}{l} \frac{\mathrm{d}x_1}{\mathrm{d}t} = a_{11}x_1 + a_{12}x_2 + f_1, \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} = a_{22}x_2 + f_2, \end{array}\right\} & t \neq n\tau + \tau_i, (i = 1, 2, \ldots, k), \\ \left.\begin{array}{l} \triangle y_1 = -d_{1i}y_1, \\ \triangle y_2 = -d_{2i}y_2, \end{array}\right\} & t = n\tau + \tau_i, (i = 1, 2, \ldots, k). \end{cases} \tag{3.5}$$

has, for every $(f_1, f_2) \in B \times B$, a unique solution $(x_1, x_2) \in B \times B$ and the operator $L : B \times B \to B \times B$ defined by $(x_1, x_2) = L(f_1, f_2)$ is linear and compact. If we define that

$$\begin{cases} \frac{\mathrm{d}x_2}{\mathrm{d}t} = a_{22}x_2 + f_2, & t \neq n\tau + \tau_i, (i = 1, 2, \ldots, k). \\ \triangle y_2 = -d_{2i}y_2, & t = n\tau + \tau_i, (i = 1, 2, \ldots, k). \end{cases}$$

has a unique solution $x_2 \in B$ and the operator $L_2 \colon B \to B$ defined by $x_2 = L_2 f_2$ is linear and compact. Furthermore, the equation

$$\begin{cases} \frac{\mathrm{d}x_1}{\mathrm{d}t} = a_{11}x_2 + f_1, & t \neq n\tau + \tau_i, (i = 1, 2, \ldots, k). \\ \triangle y_1 = -d_{1i}y_1, & t = n\tau + \tau_i, (i = 1, 2, \ldots, k). \end{cases}$$

for $f_3 \in B$ has a unique solution (since $\int_0^\tau a_{11}(s)\mathrm{d}s \neq 0$) in B and $x_1 = L_1 f_3$ defines a linear, compact operator $L_1 \colon B \to B$. Then we have

$$L(f_1, f_2) \equiv (L_1(a_{12}L_2 f_2 + f_1), L_2 f_2). \tag{3.6}$$

(b) If $\prod_{i=1}^{k}(1-d_{2i})\exp(\int_0^\tau a_{22}(s)\mathrm{d}s) = 1$, $\prod_{i=1}^{k}(1-d_{1i})\exp(\int_0^\tau a_{11}(s)\mathrm{d}s) \neq 1$, then (3.4) has exactly one independent solution in $B \times B$.

Lemma 3.3. Suppose that $a \in B$, $0 \leqslant d_i < 1, (i = 1, 2, \ldots, k)$, $\prod_{i=1}^{k}(1 - d_i)\exp(\int_0^\tau a(s)\mathrm{d}s) = 1$ and $f \in B$. Then the impulsive equation

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = ax + f, & t \neq n\tau + \tau_i, (i = 1, 2, \ldots, k), \\ \triangle x = -d_i x, & t = n\tau + \tau_i, (i = 1, 2, \ldots, k). \end{cases}$$

has a solution $x \in B$ if and only if $\int_0^\tau f(l)(exp(-\int_0^l a(s)\mathrm{d}s))\mathrm{d}l = 0$.

By the lemma 3.1, in its invariant manifold $\tilde{s} = x(t) + y(t) + z(t) = 1$, the system (1.2) reduce to a equivalently nonautonomous system as following

$$\begin{cases} \left.\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}t} &= \frac{m_1(1-y-z)y}{1+a_1(1-y-z)+b_1(1-y-z)^2} - \frac{m_2yz}{1+a_2y+b_2y^2}, \\ \frac{\mathrm{d}z}{\mathrm{d}t} &= \frac{m_2yz}{1+a_2y+b_2y^2}, \end{aligned}\right\} & \begin{array}{l} t \neq n\tau + \tau_i, \\ (i = 1, 2, \ldots, k). \end{array} \\ \left.\begin{aligned} \triangle y &= -d_i y, \\ \triangle z &= -d_i z, \end{aligned}\right\} & \begin{array}{l} t = n\tau + \tau_i, \\ (i = 1, 2, \ldots, k). \end{array} \\ y(0) > 0, z(0) \geqslant 0, y(0) + z(0) \leqslant 1, \end{cases} \tag{3.7}$$

If $m_1 > m_1^*$, for the system (3.7), by the theorem 3.1 the boundary periodic solution $(y_s(t), 0)$ is locally asymptotically stable provided with $m_2 < m_2^*$, and it is unstable provided with $m_2 > m_2^*$, hence the value m_2^* practises as a bifurcation threshold. For the system (3.7), we have the following results.

Theorem 3.2. For the system (3.7), $m_1 > m_1^*$ and $1 - b_2y_s^2(t) \geqslant 0$ $(t \in [0, \tau])$ hold, then there exists a constance $\lambda_0 > 0$, such that for each $m_2 \in (m_2^*, m_2^* + \lambda_0)$, there exists a solution $(y, z) \in B \times B$ of (3.7) satisfying $0 < y < y_s, z > 0$ and $x = 1 - y - z > 0$ for all $t > 0$. Hence, the system (1.2) has a positive τ-periodic solution $(1 - y - z, y, z)$.

Proof. Let $x_1 = y - y_s(t), x_2 = z$ in (3.7), then

$$\begin{cases} \left.\begin{aligned} \frac{\mathrm{d}x_1}{\mathrm{d}t} &= F_{11}(x_s, y_s)x_1 - F_{12}(m_2, x_s, y_s)x_2 + g_1(x_1, x_2), \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= F_{22}(m_2, y_s)x_2 + g_2(x_1, x_2), \end{aligned}\right\} & \begin{array}{l} t \neq n\tau + \tau_i \\ (i = 1, 2, \ldots, k) \end{array} \\ \left.\begin{aligned} \triangle x_1 &= -dx_1 \\ \triangle x_2 &= -dx_2 \end{aligned}\right\} & \begin{array}{l} t = n\tau + \tau_i, \\ (i = 1, 2, \ldots, k), \end{array} \end{cases} \tag{3.8}$$

where

$$\begin{aligned} F_{11}(x_s, y_s) &= \frac{m_1x_s}{1 + a_1x_s + b_1x_s^2} - \frac{m_1(1 - b_1x_s^2)y_s}{(1 + a_1x_s + b_1x_s^2)^2}, \\ F_{12}(m_2, x_s, y_s) &= \frac{m_1(1 - b_1x_s^2)y_s}{(1 + a_1x_s + b_1x_s^2)^2} + \frac{m_2y_s}{1 + a_2y_s + b_2y_s^2}, \\ F_{22}(m_2, y_s) &= \frac{m_2y_s}{1 + a_2y_s + b_2y_s^2}. \end{aligned}$$

We know that $\prod_{i=1}^{k}(1-d_i)\exp(\int_0^\tau \frac{m_2 y_s(l)}{1+a_2 y_s(l)+b_2 y_s^2(l)}\mathrm{d}l) \neq 1$, by the lemma 3.2, using L we can equivalently write the system (3.8) as the operator equation

$$(x_1, x_2) = L^*(x_1, x_2) + G(x_1, x_2), \tag{3.9}$$

where

$$\begin{aligned} L^*(x_1, x_2) &= (L_1(-F_{12}(m_2, x_s, y_s)L_2 x_2), -L_2 x_2), \\ G(x_1, x_2) &= (L_1(-F_{12}(m_2, x_s, y_s)g_2(x_1, x_2) + g_1(x_1, x_2)), L_2 g_2(x_1, x_2)). \end{aligned}$$

Here $L^*\colon B\times B \to B\times B$ is linear and compact and $G\colon B\times B \to B\times B$ is continuous and compact (since L_1 and L_2 are compact) and satisfies $G = o(|(x_1, x_2)|_0)$ near (0, 0). A nontrivial solution $(x_1, x_2) \neq (0, 0)$ for some $m_2 > 1$ yields a solution $(y, z) = (y_s + x_1, x_2)$ of the system (3.7). Solutions $(y, z) \neq (y_s, 0)$ will be called nontrivial solutions of system (3.7).

We apply well-known local bifurcation techniques to (3.9). As is well known, bifurcation can occur only at the nontrivial solution of the linearized problem

$$(y_1, y_2) = L^*(y_1, y_2), \quad m_2 > 0. \tag{3.10}$$

If $(y_1, y_2) \in B \times B$ is a solution of (3.10) for some $m_2 > 0$, then by the very manner in which L^* was defined, (y_1, y_2) solves the system

$$\begin{cases} \left.\begin{aligned} \frac{\mathrm{d}x_1}{\mathrm{d}t} &= F_{11}(x_s, y_s)x_1 - F_{12}(m_2, x_s, y_s)x_2, \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= F_{22}(m_2, y_s)x_2, \end{aligned}\right\} & t \neq n\tau + \tau_i, (i = 1, 2, \ldots, k), \\ \left.\begin{aligned} \triangle x_1 &= -d_i x_1, \\ \triangle x_2 &= -d_i x_2, \end{aligned}\right\} & t = n\tau + \tau_i, (i = 1, 2, \ldots, k). \end{cases} \tag{3.11}$$

and conversely. Using Lemma 3.2 (b), we see that (3.11) and hence (3.10) has one nontrivial solution in $B \times B$ if and only if $m_2 = m_2^*$. Hence there exists a continuum $C = \{(m_2; x_1, x_2)\} \subseteq (0, \infty) \times B \times B$ nontrivial solutions of (3.10) such that the closure $\bar{C}$ contains $(m_2^*; 0, 0)$. This continuum gives rise to a continuum $C_1 = \{(m_2; y, z)\} \subseteq (0, \infty) \times B \times B$ of the solutions of (3.7) whose closure $\bar{C}_1$ contains the bifurcation point $(m_2^*; y_s, 0)$.

To see that solutions in C_1 correspond to solutions (y, z) of (3.7), we investigate the nature of the continuum C near the bifurcation point $(m_2^*; 0, 0)$ by expending m_2 and (x_1, x_2) in Lyapunov-Schmidt series:

$$\begin{aligned} m_2 &= m_2^* + \lambda\varepsilon + \cdots, \\ x_1 &= x_{11}\varepsilon + x_{12}\varepsilon^2 + \cdots, \\ x_2 &= x_{21}\varepsilon + x_{22}\varepsilon^2 + \cdots, \end{aligned}$$

for $x_{ij} \in B$ where ε is a small parameter. If we substitute these series into the differential system (3.7) and equate coefficients of ε and ε^2 we find that

$$\begin{cases} \left.\begin{aligned} \frac{\mathrm{d}x_{11}}{\mathrm{d}t} &= F_{11}(x_s, y_s)x_{11} - F_{12}(m_2, x_s, y_s)x_{21}, \\ \frac{\mathrm{d}x_{21}}{\mathrm{d}t} &= F_{22}(m_2, y_s)x_{21}, \end{aligned}\right\} & t \neq n\tau + \tau_i, \ (i = 1, 2, \ldots, k), \\ \left.\begin{aligned} \triangle x_{11} &= -d_i x_{11}, \\ \triangle x_{21} &= -d_i x_{21}, \end{aligned}\right\} & t = n\tau + \tau_i, \ (i = 1, 2, \ldots, k), \end{cases} \tag{3.12}$$

and

$$\begin{cases} \left.\begin{aligned} x'_{12} &= F_{11}(x_s, y_s)x_{12} - F_{12}(m_2^*, x_s, y_s)x_{22} + G_{12}(x_{11}, x_{21}, \lambda), \\ x'_{22} &= F_{22}(m_2^*, y_s)x_{22} + \frac{x_{21}}{a_2+y_s+b_2y_s^2}\left(\lambda y_s + \frac{m_2^*(1-b_2y_s^2)x_{11}}{1+a_2y_s+b_2y_s^2}\right), \end{aligned}\right\} & t \neq n\tau + \tau_i, \ (i = 1, 2, \ldots, k), \\ \left.\begin{aligned} \triangle x_{12} &= -d_i x_{12}, \\ \triangle x_{22} &= -d_i x_{22}, \end{aligned}\right\} & t = n\tau + \tau_i, \ (i = 1, 2, \ldots, k), \end{cases}$$

respectively. Thus, $(x_{11}, x_{21}) \in B \times B$ must be a solution of (3.10). We choose the specific solution satisfying the initial conditions $x_{21}(0) = 1$. Then

$$\begin{aligned} x_{21} = \prod_{j=0}^{i-1}(1-d_j)exp\left(\int_{n\tau}^{t}(\frac{m_2^*y_s(l)}{1+a_2y_s(l)+b_2y_s^2(l)})\mathrm{d}l\right) > 0, \\ n\tau + \tau_{i-1} < t \leqslant n\tau + \tau_i, \\ x_{21}(0) = 1. \end{aligned}$$

Moreover $x_{11} < 0$ for all t, (since $m_1 > m_1^*$ and (2.11), hence $\int_0^\tau \frac{m_1x_s(1-b_1x_s^2)}{1+a_1x_s+b_1x_s^2}\mathrm{d}l < 0$, which implies that the Green's function for first equation in (3.11) is positive). Using Lemma 3.3, we find that

$$\lambda = -\frac{\int_0^\tau \frac{m_2^*x_{11}(t)x_{21}(t)(1-b_2y_s^2(t))}{(1+a_2y_s(t)+b_2y_s^2(t))^2}\exp\left(-\int_0^t \frac{m_2^*y_s(l)}{1+a_2y_s(t)+b_2y_s^2(t)}\mathrm{d}l\right)\mathrm{d}t}{\int_0^\tau \frac{x_{21}y_s}{1+a_2y_s(t)+b_2y_s^2(t)}\exp\left(-\int_0^t \frac{m_2^*y_s(l)}{1+a_2y_s(t)+b_2y_s^2(t)}\mathrm{d}l\right)\mathrm{d}t} > 0,$$

provided with $1 - b_2y_s^2(t) \geqslant 0$. Thus we see that near the bifurcation point $(m_2^*; 0, 0)$ (say, for $0 < |m_2 - m_2^*| = \lambda|\varepsilon| < \lambda_0$) the continuum C has two (subcontinua) branches corresponding to $\varepsilon < 0, \varepsilon > 0$ respectively:

$$C^+ = \{(m_2; x_1, x_2) : m_2^* < m_2 < m_2^* + \lambda_0, x_1 < 0, x_2 > 0\},$$

$$C^- = \{(m_2; x_1, x_2) : m_2^* - \lambda_0 < m_2 < m_2^*, x_1 > 0, x_2 < 0\}.$$

The solution is on C^+ which prove the theorem, since $\lambda > 0$ is equivalent to $m_2 > m_2^*$. We have left only to show that $y = x_1 + y_s > 0$ for all t. This is easy, for if λ_0 is small, then y is near y_s in the sup norm of B; thus since y_s is bounded away from zero, so is y. At same time, by theorem 3.1, for the system

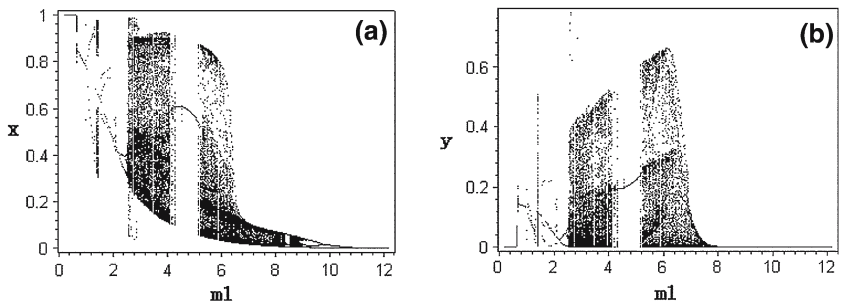

Figure 1. Bifurcation diagrams of system (1.2) with $m_2 = 8, a_1 = 0.2, a_2 = 0.3, b_1 = 0.1, b_2 = 0.3, d_1 = 0.9, d_2 = 0.8, \tau_1 = 0.4\tau, \tau_2 = \tau = 8$ and $0.2 < m_1 \leqslant 12.2$ and initial values $x_0 = 0.5, y_0 = 0.7, z_0 = 0.05$.

(1.2), y is near y_s means that x is near x_s; thus $x = 1-y-z > 0$. We notice that the periodic solution (y, z) is τ-periodic. So $x = 1-y-z$ is piecewise continuous and τ-periodic. We complete the proof. □

4. Chemostat chaos

In this section, we will analyze the complexity of the impulsive system (1.2). By theorem 2.1, 3.1 and 3.2, we know that if $m_1 < m_1^*$, the periodic solution $(\tilde{s}(t), 0, 0)$ is globally asymptotically stable; if $m_1 > m_1^*$ and $m_2 < m_2^*$, then the $(x_s(t), y_s(t), 0)$ is globally asymptotically stable. According to Theorem 3.2, if $m_1 > m_1^*$ and $m_2 > m_2^*$, the predator begins to invade the system.

We want to investigate the influence of m_1. In the system (1.2), set $m_2 = 8$, $a_1 = 0.2, a_2 = 0.3, b_1 = 0.1, b_2 = 0.3, d_1 = 0.9, d_2 = 0.8, \tau_1 = 0.4\tau, \tau_2 = \tau = 8$ and $0.2 < m_1 \leqslant 12.2$. The influences of m_1 may be documented by stroboscopically sampling some of the variables over a range of m_1 values. We numerically integrated system (1.2) for 500 pulsing cycles at each value of m_1. For each m_1, we plotted the last 200 measures of the prey y and the predator z. Since we sampled at the forcing period, periodic solutions of period τ appear as fixed points, periodic solutions of period 2τ appear as two cycles, and so forth. The resulting bifurcation diagrams (figure 1) clear show that: with increasing m_1 from 0.2 to 12.2, the system experiences process of cycles $\rightarrow$ periodic doubling cascade $\rightarrow$chaos $\rightarrow$ cycles $\rightarrow$ periodic doubling cascade $\rightarrow$chaos $\rightarrow$ periodic halfing cascade $\rightarrow$ cycles, which is characterized by (1) period doubling, (2) period halfing.

When m_1 is small ($m_1 < q_0 \approx 0.66$), the solution $(1, 0, 0)$ is stable. When $m_1 > q_0$, the prey begins invade the system and the solution $(x_s, y_s, 0)$ is stable if $m_1 < q_1 (> q_0)$. When $m_1 > q_1$, the predator begins invade and a stable positive period solution (figure 2(a))is bifurcated from $(x_s, y_s, 0)$ if $m_1 < q_2 \approx 0.9$. However, when $m_1 > q_2$, the stability of τ-periodic solution is destroyed and 2τ-periodic solution occurs (figure 2(b)) and is stable if $m_1 < q_3 \approx 1.26$. When

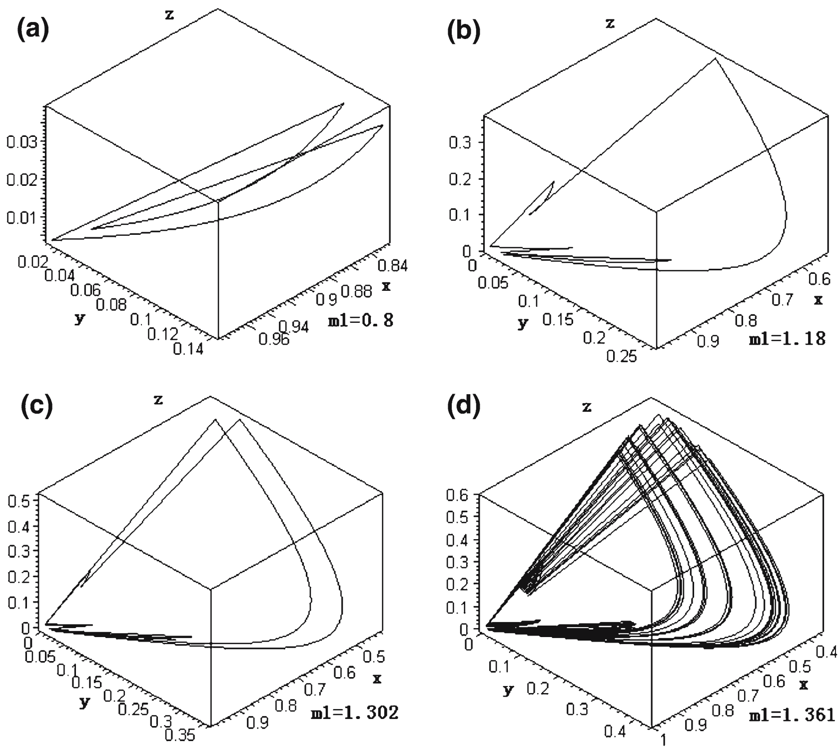

Figure 2. Doubling bifurcation. (a)–(d) Phase portraits of τ, 2τ, 4τ-period solutions and chaotic solution for $m1$= 0.8, 1.18, 1.302 and 1.361, respectively.

$m_1 > q_4 \approx 1.32$, it is unstable and there is a cascade of period doubling bifurcations leading to chaos (figure 2 (c, d)). Continuously increasing $m_1 \approx 2.2$, the chaotic solution suddenly shrinks to a τ-period solution and further the system shows next doubling bifurcations. A typical chaotic oscillation is captured when $m_1 = 2.98$ (figure 3). When $m_1 > 5.18$ is followed by a cascade of periodic halfing bifurcations from chaos to cycles (figure 4). This periodic-doubling route to chaos is the hallmark of the logistic and Ricker maps [19, 20] and has been studied extensively by Mathematicians [21]. Periodic halving is the flip bifurcation in the opposite direction, which is also observed in [22].

It is obvious, from the resulting bifurcation diagrams figure 1, we observe that there exists more than one attractor for the same m_1. In such a case, the state that the system will reach depends on its initial value. An example of two different stable states, a τ-period attractor and a strange attractor, observed for the same values of impulsive period, is shown in figure 5.

We want to investigate the influence of m_2. Set $m_1 = 6, a_1 = 0.2, a_2 = 0.2, b_1 = 0.1, b_2 = 0.3, d_1 = 0.9, d_2 = 0.8, \tau_1 = 0.4\tau, \tau_2 = \tau = 8$ and $1 < m_2 \leqslant 18.8$. We numerically integrated system (1.2) for 500 pulsing cycles

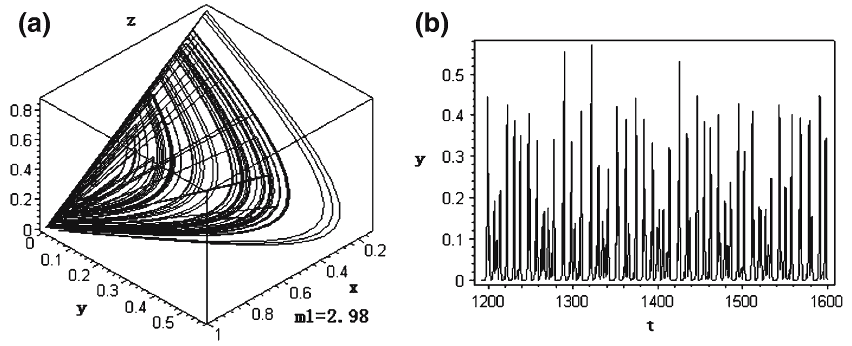

Figure 3. A strange attractor: (a) phase portrait of system (1.2) of m1=2.98, (b) time series of y solution with initial values $x_0 = 0.5, y_0 = 0.7, z_0 = 0.05$.

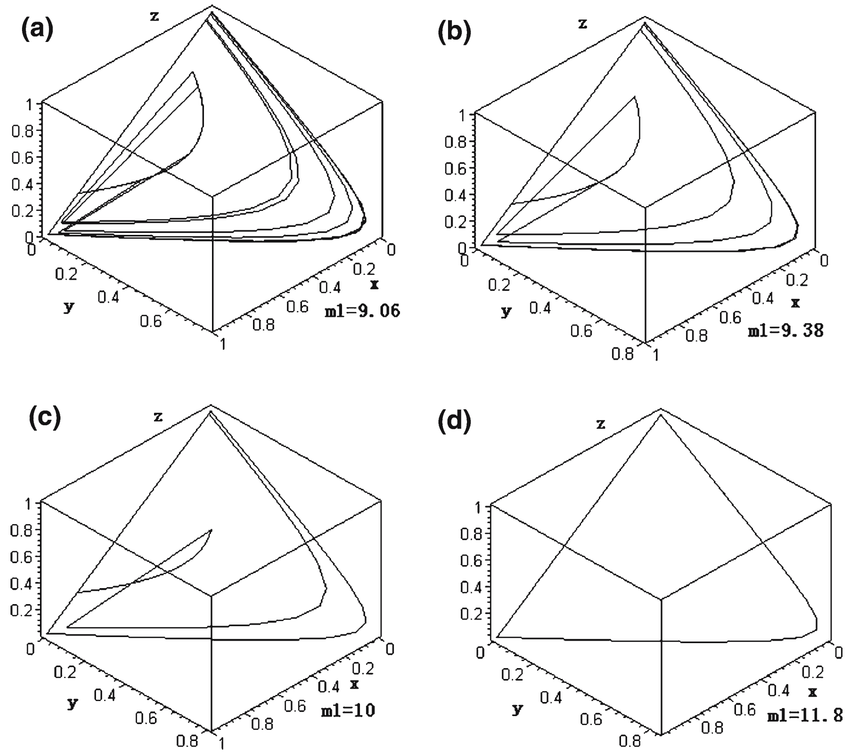

Figure 4. Halving bifurcation. (a)–(d) Phase portraits of 6τ, 3τ, 2τ and τ-period solutions for $m1 = 9.06$, 9.38, 10 and 11.8, respectively.

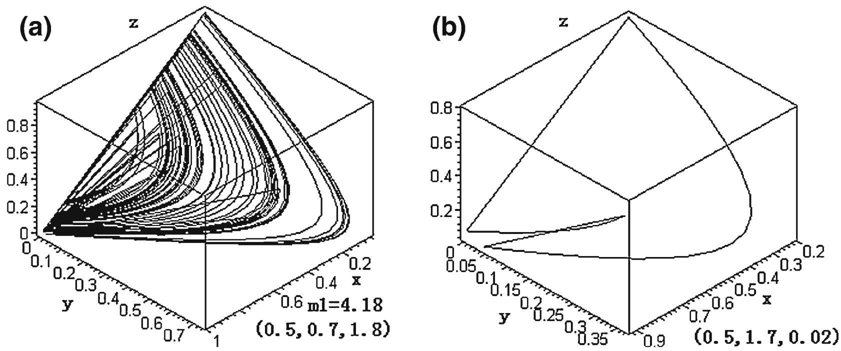

Figure 5. Coexistence of a strange attractor with a 2τ-periodic solution when $m_1 = 4.18$: (a) a strange attractor with (x(0),y(0),z(0)) = (0.5, 0.7, 1.8), (b) a τ-periodic solution with (x(0),y(0), z(0) =(0.5, 1.7, 0.02).

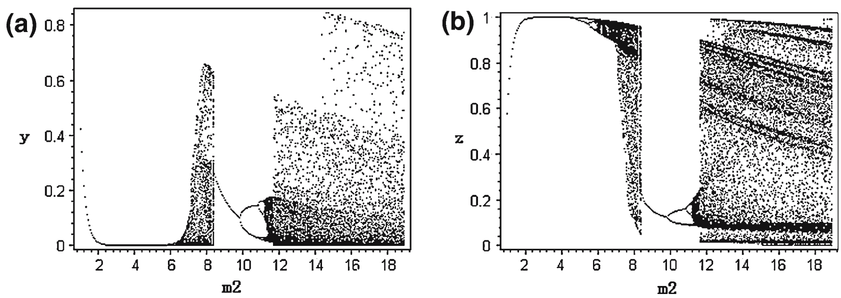

Figure 6. Bifurcation diagrams of system (1.2) with $m_1 = 6, a_1 = 0.2, a_2 = 0.2, b_1 = 0.1, b_2 = 0.3, d_1 = 0.9, d_2 = 0.8, \tau_1 = 0.4\tau, \tau_2 = \tau = 8$ and $1 < m_2 \leqslant 18.8$ and initial values $x_0 = 0.5, y_0 = 0.7, z_0 = 0.05$.

at each value of m_2. For each m_2, we plotted the last 200 stroboscopic measures of the prey y and the predator z. The resulting bifurcation diagrams (figure 6) clear show that: with increasing m_2 from 0.2 to 18.8, the system experiences process two time's periodic doubling bifurcations. Comparable changes occur with an increase in the pulse period τ. Set $m_1 = 6, m_2 = 8, a_1 = 0.2, a_2 = 0.2, b_1 = 0.1, b_2 = 0.3, d_1 = 0.9, d_2 = 0.8, \tau_1 = 0.4\tau, \tau_2 = \tau$ and $0.2 < \tau \leqslant 18.2$. The resulting bifurcation diagrams (figure 7) clear show that: with increasing τ from 0.2 to 18.2, the system also experiences process two time's periodic doubling bifurcations.

Pitchfork bifurcations and tangent (saddle node) bifurcations are abundantly evident in cycles in figures 1, 6 and 7, as well as attractor crises (the phenomenon of “crisis" in which chaotic attractors suddenly appear or disappear, or change size discontinuously as or change size discontinuously as a parameter smoothly varies, was first extensively analyzed by Grebogi et al. [23]). For

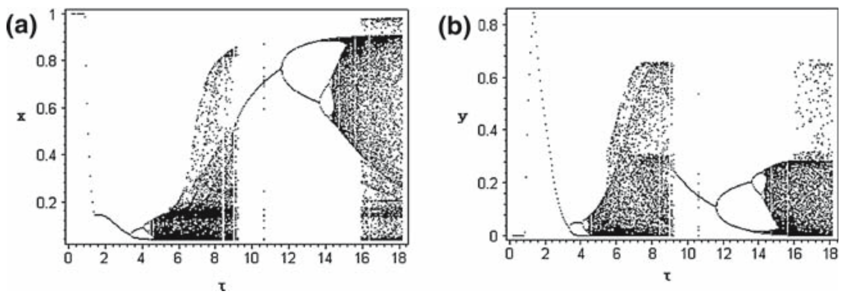

Figure 7. Bifurcation diagrams of system (1.2) with $m_1 = 6, m_2 = 8, a_1 = 0.2, a_2 = 0.2, b_1 = 0.1, b_2 = 0.3, d_1 = 0.9, d_2 = 0.8, \tau_1 = 0.4\tau, \tau_2 = \tau$ and $0.2 < \tau \leqslant 18.2$ and initial values $x_0 = 0.5, y_0 = 0.7, z_0 = 0.05$.

instance, in figure 7, when the forcing period τ is slightly increased beyond $\tau = 9.17$, the chaotic attractor abruptly disappears, thus constituting a type of crisis.

5. Conclusions

In this paper, we introduce and study a model of a predator-prey system with Monod–Haldene type functional response with seasonally variably pulsed input and washout. Firstly we find the invasion threshold of the prey, which is $m_1^* := \frac{-(1+a_1+b_1)\sum_{i=1}^{k}\ln(1-d_i)}{\tau}$. If $m_1 < m_1^*$, the periodic periodic solution $(1, 0, 0)$ is globally asymptotically stable and if $m_1 > m_1^*$, the prey starts to invade the system. Furthermore, by using Floquet theorem and small amplitude perturbation skills, we have proved that if $m_1 > m_1^*$, there exists $m_2^* := \frac{-\sum_{i=1}^{k}\ln(1-d_i)}{\int_0^\tau \frac{y_s(l)}{1+a_2 y_s(l)+b_2 y_s^2(l)}\mathrm{d}l}$ to play as the invasion threshold of the predator, that is to say, if $m_2 < m_2^*$ the boundary solution $(x_s, y_s, 0)$ is globally asymptotically stable and if $m_2 > m_2^*$ the solution $(x_s, y_s, 0)$ is unstable. By using standard techniques of bifurcation theory, we prove that above this threshold there are periodic oscillations in substrate, prey and predator.

Choosing different coefficients m_1, m_2 and pulsed period τ as bifurcation parameters, we have obtained bifurcation diagrams (figure 1, 6, 7). Bifurcation diagrams have shown that there exists complexity for system (1.2) including periodic doubling cascade, periodic halving cascade and Pitchfork bifurcations and tangent bifurcations. Further, we can conclude from the results obtained in the paper that periodically pulsed input and washout make the food chain chemostat occur with various kinds of periodic fluctuations, period-one attractors, multi-period attractors and chaotic attractors. More than one stable state may exist for the same parameter values. All these results show that dynamical behavior of system (1.2) becomes more complex under periodically impulsive input and washout.

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